

# Towards Theory of Massive-Parallel Proofs. Cellular Automata Approach

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## Abstract

In the paper I sketch a theory of massively parallel proofs using cellular automata presentation of deduction. In this presentation inference rules play the role of cellular-automatic local transition functions. In this approach we completely avoid axioms as necessary notion of deduction theory and therefore we can use cyclic proofs without additional problems. As a result, a theory of massive-parallel proofs within unconventional computing is proposed for the first time.

## 1 Introduction

Non-well-founded proofs including cyclic proofs have been actively studying recently (see [5] – [7], [11]). Their features consist in that in the classical theory of deduction, derivation trees, on the one hand, are finite and, on the other hand, they are without cycles, while in the non-well-founded approach they can be infinite and, at the same time, circles occur in them. Non-well-founded proofs have different applications in computer science. In the paper I am proposing a more radical approach than other non-well-founded approaches to deduction by defining massive-parallel proofs and rejecting axioms in proof theory. This novel approach is characterized as follows:

- Deduction is considered as a transition in cellular automata, where states of cells are regarded as well-formed formulas of a logical language.
- We build up derivations without using axioms, therefore there is no sense in distinguishing logic and theory (i.e. logical and nonlogical axioms), derivable and provable formulas, etc.

- In deduction we do not obtain derivation trees and instead of the latter we find out derivation traces, i.e. a linear evolution of each singular premise.
- Some derivation traces are circular, i.e. some premises are derivable from themselves.
- Some derivation traces are infinite.

## 2 Proof-theoretic cellular automata

For any logical language  $\mathcal{L}$  we can construct a *proof-theoretic cellular automaton* (instead of conventional deductive systems) simulating massive-parallel proofs.

**Definition 1** A *proof-theoretic cellular automaton* is a 4-tuple  $\mathcal{A} = \langle \mathbb{Z}^d, S, N, \delta \rangle$ , where

- $d \in \mathbb{N}$  is a number of dimensions and the members of  $\mathbb{Z}^d$  are referred as cells,
- $S$  is a finite or infinite set of elements called the states of an automaton  $\mathcal{A}$ , the members of  $\mathbb{Z}^d$  take their values in  $S$ , the set  $S$  is collected from well-formed formulas of a language  $\mathcal{L}$ .
- $N \subset \mathbb{Z}^d \setminus \{0\}^d$  is a finite ordered set of  $n$  elements,  $N$  is said to be a neighborhood,
- $\delta: S^{n+1} \rightarrow S$  that is  $\delta$  is the inference rule of a language  $\mathcal{L}$ , it plays the role of local transition function of an automaton  $\mathcal{A}$ .

As we see an automaton is considered on the endless  $d$ -dimensional space of integers, i.e. on  $\mathbb{Z}^d$ . Discrete time is introduced for  $t = 0, 1, 2, \dots$  fixing each step of inferring.

For any given  $z \in \mathbb{Z}^d$ , its neighborhood is determined by  $z + N = \{z + \alpha : \alpha \in N\}$ . There are two often-used neighborhoods:

- Von Neumann neighborhood  $N_{VN} = \{z \in \mathbb{Z}^d : \sum_{k=1}^d |z_k| = 1\}$
- Moore neighborhood  $N_M = \{z \in \mathbb{Z}^d : \max_{k=\overline{1,d}} |z_k| = 1\} = \{-1, 0, 1\}^d \setminus \{0\}^d$

For example, if  $d = 2$ ,  $N_{VN} = \{(-1, 0), (1, 0), (0, -1), (0, 1)\}$ ;  $N_M = \{(-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 1), (1, -1), (1, 0), (1, 1)\}$ .

In the case  $d = 1$ , von Neumann and Moore neighborhoods coincide. It is easily seen that  $|N_{VN}| = 2d$ ,  $|N_M| = 3^d - 1$ .

At the moment  $t$ , the *configuration* of the whole system (or the *global state*) is given by the mapping  $x^t: \mathbb{Z}^d \rightarrow S$ , and the *evolution* is the sequence  $x^0, x^1, x^2, \dots$  defined as follows:  $x^{t+1}(z) = \delta(x^t(z), x^t(z + \alpha_1), \dots, x^t(z + \alpha_n))$ , where  $\langle \alpha_1, \dots, \alpha_n \rangle \in N$ . Here  $x^0$  is the initial configuration, and it fully determines the future behavior of the automaton. It is the set of all premises (not axioms).

We assume that  $\delta$  is an inference rule, i.e. a mapping from the set of premises (their number cannot exceed  $n = |N|$ ) to a conclusion. For any  $z \in \mathbb{Z}^d$  the sequence  $x^0(z), x^1(z), \dots, x^t(z), \dots$  is called a *derivation trace from a state*  $x^0(z)$ . If there exists  $t$  such that  $x^t(z) = x^l(z)$  for all  $l > t$ , then a derivation trace is *finite*. It is *circular/cyclic* if there exists  $l$  such that  $x^t(z) = x^{t+l}(z)$  for all  $t$ .

**Definition 2** *In case all derivation traces of a proof-theoretic cellular automaton  $\mathcal{A}$  are circular, this automaton  $\mathcal{A}$  is said to be reversible.*

Notice that  $x^{t+1}$  depends only upon  $x^t$ , i.e. the previous configuration. It enables us to build the function  $G_{\mathcal{A}}: C_{\mathcal{A}} \rightarrow C_{\mathcal{A}}$ , where  $C_{\mathcal{A}}$  is the set of all possible configurations of the cellular automaton  $\mathcal{A}$  (it is the set of all mappings  $\mathbb{Z}^d \rightarrow S$ , because we can take each element of this set as the initial configuration  $x^0$ , though not every element can arise in the evolution of some other configuration).  $G_{\mathcal{A}}$  is called the *global function* of the automaton.

**Example 1 (modus ponens)** *Consider a propositional language  $\mathcal{L}$  that is built in the standard way with the only binary operation of implication  $\supset$ . Let us suppose that well-formed formulas of that language are used as the set of states for a proof-theoretic cellular automaton  $\mathcal{A}$ . Further, assume that modus ponens is a transition rule of this automaton  $\mathcal{A}$  and it is formulated for any  $\varphi, \psi \in \mathcal{L}$  as follows:*

$$x^{t+1}(z) = \begin{cases} \psi, & \text{if } x^t(z) = \varphi \supset \psi \text{ and } \varphi \in (z + N); \\ x^t(z), & \text{otherwise.} \end{cases}$$

*The further dynamics will depend on the neighborhood. If we assume the Moor neighborhood in the 2-dimensional space, this dynamics will be exemplified by the evolution of cell states in Fig. 1 – Fig. 3.*

This example shows that first we completely avoid axioms and secondly we take premisses from the cell states of the neighborhood according to a

$(p \supset q) \supset r$	$p \supset (p \supset q)$	$p \supset q$	$(p \supset q) \supset (p \supset q)$	$(r \supset p) \supset r$
$(p \supset r) \supset (q \supset r)$	$p \supset q$	$p$	$p \supset (p \supset q)$	$r \supset p$
$p \supset r$	$p$	$p \supset (q \supset (p \supset q))$	$p$	$r$
$p \supset (q \supset r)$	$p \supset p$	$p \supset q$	$(p \supset r) \supset (q \supset p)$	$p \supset r$
$p \supset q$	$p \supset (q \supset p)$	$q$	$p \supset r$	$p$

Figure 1: An initial configuration of a proof-theoretic cellular automaton  $\mathcal{A}$  with the Moor neighborhood in the 2-dimensional space, its states run over formulas set up in a propositional language  $\mathcal{L}$  with the only binary operation  $\supset$ ,  $t = 0$ . Notice that  $p, q, r$  are propositional variables.

$r$	$p \supset q$	$q$	$p \supset q$	$r$
$q \supset r$	$q$	$p$	$p \supset q$	$p$
$r$	$p$	$q \supset (p \supset q)$	$p$	$r$
$q \supset r$	$p$	$q$	$q \supset p$	$r$
$p \supset q$	$p \supset (q \supset p)$	$q$	$r$	$p$

Figure 2: An evolution of  $\mathcal{A}$  described in Fig. 1 at the time step  $t = 1$ .

transition function. As a result, we do not come across proof trees in our novel approach to deduction taking into account that a cell state has just a linear dynamics (the number of cells and their location do not change). This allows us evidently to simplify deductive systems.

Now we are trying to consider a cellular-automaton presentation of two basic deductive approaches: Hilbert's type and sequent ones.

**Example 2 (Hilbert's inference rules)** *Suppose a propositional language  $\mathcal{L}$  contains two basic propositional operations: negation and disjunction. As usual, the set of all formulas of  $\mathcal{L}$  is regarded as the set of states of an appropriate proof-theoretic cellular automata. In that we will use the exclusive*

$r$	$q$	$q$	$q$	$r$
$r$	$q$	$p$	$q$	$p$
$r$	$p$	$q$	$p$	$r$
$r$	$p$	$q$	$p$	$r$
$q$	$p$	$q$	$r$	$p$

Figure 3: An evolution of  $\mathcal{A}$  described in Fig. 1 at the time step  $t = 3$ . Its configuration cannot vary further.

disjunction of the following five inference rules converted from Joseph R. Shoenfield's deductive system:

$$x^{t+1}(z) = \begin{cases} \psi \vee \varphi, & \text{if } x^t(z) = \varphi; \\ \varphi, & \text{if } x^t(z) = \varphi \vee \varphi; \\ (\chi \vee \psi) \vee \varphi, & \text{if } x^t(z) = \chi \vee (\psi \vee \varphi); \\ \chi \vee \psi, & \text{if } x^t(z) = \varphi \vee \chi \text{ and } (\neg\varphi \vee \psi) \in (z + N); \\ \chi \vee \psi, & \text{if } x^t(z) = \neg\varphi \vee \psi \text{ and } (\varphi \vee \chi) \in (z + N). \end{cases}$$

**Example 3 (sequent inference rules)** Let us take a sequent propositional language  $\mathcal{L}$ , in which the classical propositional language with negation, conjunction, disjunction and implication is extended by adding the sequent relation  $\hookrightarrow$ . Recall that a sequent is an expression of the form  $\Gamma_1 \hookrightarrow \Gamma_2$ , where  $\Gamma_1 = \{\varphi_1, \dots, \varphi_j\}$ ,  $\Gamma_2 = \{\psi_1, \dots, \psi_i\}$  are finite sets of well-formed formulas of the standard propositional language, that has the following interpretation:  $\Gamma_1 \hookrightarrow \Gamma_2$  is logically valid iff

$$\bigwedge_j \varphi_j \supset \bigvee_i \psi_i$$

is logically valid. Let  $S$  denote the set of all sequents of  $\mathcal{L}$ , furthermore let us assume that this family  $S$  is regarded as the set of states for a proof-theoretic cellular automaton  $\mathcal{A}$ . The transition rule of  $\mathcal{A}$  is an exclusive disjunction of the 14 singular rules (6 structural rules and 8 logical rules):

$$x^{t+1}(z) = \Gamma_1 \hookrightarrow \Gamma_2, \begin{cases} \text{if } \Gamma_1 \hookrightarrow \Gamma_2 \text{ is a result of applying to } x^t(z) \\ \text{either one of structural rules} \\ \text{or the left (right) introduction of negation} \\ \text{or the left introduction of conjunction} \\ \text{or the right introduction of disjunction} \\ \text{or the right introduction of implication.} \end{cases}$$

$$x^{t+1}(z) = \begin{cases} \Gamma \hookrightarrow \Gamma', \psi \wedge \chi, & \text{if } x^t(z) = \Gamma \hookrightarrow \Gamma', \psi \text{ and } (\Gamma \hookrightarrow \Gamma', \chi) \in (z + N); \\ \Gamma, \psi \vee \chi \hookrightarrow \Gamma', & \text{if } x^t(z) = \Gamma, \psi \hookrightarrow \Gamma' \text{ and } (\Gamma, \chi \hookrightarrow \Gamma') \in (z + N); \\ \psi \supset \chi, \Gamma, \Delta \hookrightarrow \Gamma', \Delta', & \text{if } x^t(z) = \Gamma \hookrightarrow \Gamma', \psi \text{ and } (\chi, \Delta \hookrightarrow \Delta') \in (z + N). \end{cases}$$

$\frac{\Gamma \hookrightarrow N(x)}{\Gamma \hookrightarrow N(x+1)},$	$\frac{\Gamma \hookrightarrow \Delta}{\Gamma \hookrightarrow \Delta, N(0)},$	$\frac{\Gamma \hookrightarrow E(x)}{\Gamma \hookrightarrow O(x+1)},$	$\frac{\Gamma \hookrightarrow O(x)}{\Gamma \hookrightarrow E(x+1)},$	$\frac{\Gamma \hookrightarrow \Delta}{\Gamma \hookrightarrow \Delta, E(0)},$
$\frac{N(x) \hookrightarrow \Delta}{N(x+1) \hookrightarrow \Delta},$	$\frac{\Gamma \hookrightarrow \Delta}{\Gamma, N(0) \hookrightarrow \Delta},$	$\frac{E(x) \hookrightarrow \Delta}{O(x+1) \hookrightarrow \Delta},$	$\frac{O(x) \hookrightarrow \Delta}{E(x+1) \hookrightarrow \Delta},$	$\frac{\Gamma \hookrightarrow \Delta}{\Gamma, E(0) \hookrightarrow \Delta},$
$\frac{\Gamma, t=0 \hookrightarrow \Delta \quad \Gamma, t=x+1, N(x) \hookrightarrow \Delta}{\Gamma, N(t) \hookrightarrow \Delta} \text{ (Case } N), \text{ where } x \notin FV(\Gamma \cup \Delta \cup \{N(t)\}),$				
$\frac{\Gamma \hookrightarrow \Delta}{\Gamma[x] \hookrightarrow \Delta[x]} \text{ (Substitution).}$				

Figure 4: Inference rules for predicates  $N$  ('being a natural number'),  $E$  ('being an even number'),  $O$  ('being an odd number'), see [5].

**Example 4 (Brotherston's cyclic proofs)** *The sequent language used in the previous example we extend by adding predicates  $N$ ,  $E$ ,  $O$  and appropriate inference rules of Fig. 4 for them. Further, let us extend also the automaton of Example 3 in the same way by representing inference rules of Fig. 4 in the cellular-automatic form.*

*Now we assume that a cell has an initial state  $[\Gamma, N(z) \hookrightarrow \Delta, O(z), E(z)]$  and its neighbor cell an initial state  $[\Gamma, z = 0 \hookrightarrow \Delta]$  that is equal to  $[\Gamma, z = 0 \hookrightarrow \Delta, O(z), E(z)]$  for any  $t = 4, 14, 24, \dots$  and to  $[\Gamma, z = 0 \hookrightarrow \Delta, E(z), O(z)]$  for any  $t = 9, 19, 29, \dots$ . Then we will have the following infinite cycle:*

$$\begin{aligned}
& [\Gamma, N(z) \hookrightarrow \Delta, O(z), E(z)] \xrightarrow{(\text{substitution})} [\Gamma, N(y) \hookrightarrow \Delta, O(y), E(y)] \\
& \longrightarrow [\Gamma, N(y) \hookrightarrow \Delta, O(y), O(y+1)] \longrightarrow [\Gamma, N(y) \hookrightarrow \Delta, E(y+1), O(y+1)] \\
& \longrightarrow [\Gamma, z = (y+1), N(y) \hookrightarrow \Delta, O(z), E(z)] \xrightarrow{(\text{case } N)} [\Gamma, N(z) \hookrightarrow \Delta, E(z), O(z)] \longrightarrow \dots
\end{aligned}$$

Another instance of cyclic proof is given in Example 5. As we see, the possibility of cyclic derivation traces depends on configuration of cell states.

Traditional tasks concerning proof theory like completeness and independence of axioms lose their sense in massive-parallel proof theory, although it can be readily shown that we can speak about consistency:

**Proposition 1** *Proof theories given in Examples 2 and 3 are consistent, i.e. we cannot deduce a contradiction within them.*

### 3 The proof-theoretic cellular automaton for Belousov-Zhabotinsky reaction

Massive-parallel computing is observed everywhere in natural systems. There are different approaches to nature-inspired computing: reaction-diffusion computing [1] – [3], [12], chemical computing [4], biological computing [8], [10], etc. In all those computational models parallel inferring and concurrency are assumed as key notions. In the paper [9] a hypothesis was put forward that the paradigm of parallel and concurrent computation caused by rejecting the set-theoretic axiom of foundation can be widely applied in modern physics. In this section we are analyzing simulating Belousov-Zhabotinsky reaction within the framework of our theory of massive-parallel proofs.

Let us consider a proof-theoretic cellular automaton with circular proofs for the Belousov-Zhabotinsky reaction containing feedback relations. The mechanism of this reaction (namely cerium(III)  $\longleftrightarrow$  cerium(IV) catalyzed reaction) is very complicated: its recent model contains 80 elementary steps and 26 variable species concentrations. Let us consider a simplification of Belousov-Zhabotinsky reaction assuming that the set of states consists just of the following reactants:  $Ce^{3+}$ ,  $HBrO_2$ ,  $BrO_3^-$ ,  $H^+$ ,  $Ce^{4+}$ ,  $H_2O$ ,  $BrCH(COOH)_2$ ,  $Br^-$ ,  $HCOOH$ ,  $CO_2$ ,  $HOBr$ ,  $Br_2$ ,  $CH_2(COOH)_2$  which interact according to inference rules (reactions) (1) – (7). In this reaction we observe sudden oscillations in color from yellow to colorless, allowing the oscillations to be observed visually. In spatially nonhomogeneous systems (such as a simple petri dish), the oscillations propagate as spiral wave fronts. The oscillations last about one minute and are repeated over a long period of time. The color changes are caused by alternating oxidation-reductions in which cerium changes its oxidation state from cerium(III) to cerium(IV) and vice versa:  $Ce^{3+} \longrightarrow Ce^{4+} \longrightarrow Ce^{3+} \longrightarrow \dots$

When  $Br^-$  has been significantly lowered, the reaction pictured by inference rule (1) causes an exponential increase in bromous acid ( $HBrO_2$ ) and the oxidized form of the metal ion catalyst and indicator, cerium(IV). Bromous acid is subsequently converted to bromate ( $BrO_3^-$ ) and  $HOBr$  (the step (3)). Meanwhile, the step (2) reduces the cerium(IV) to cerium(III) and simultaneously increase bromide ( $Br^-$ ) concentration. Once the bromide concentration is high enough, it reacts with bromate ( $BrO_3^-$ ) and  $HOBr$  in (4) and (6) to form  $Br_2$ , further  $Br_2$  reacts with  $CH_2(COOH)_2$  to form  $BrCH(COOH)_2$  and the process begins again. Thus, parallel processes in (1) – (7) have several cycles which are performed synchronously.

The proof-theoretic simulation of Belousov-Zhabotinsky reaction can be defined as follows:

**Definition 3** Consider a propositional language  $\mathcal{L}$  with the only binary operation  $\oplus$ , it is built in the standard way over the set of variables  $S = \{Ce^{3+}, HBrO_2, BrO_3^-, H^+, Ce^{4+}, H_2O, BrCH(COOH)_2, Br^-, HCOOH, CO_2, HOBr, Br_2, CH_2(COOH)_2\}$ . Let  $S$  be the set of states of proof-theoretic cellular automaton  $\mathcal{A}$ . The inference rule of the automaton is presented by the conjunction of singular inference rules (1) – (7):

$$(1) \wedge (2) \wedge (3) \wedge (4) \wedge (5) \wedge (6) \wedge (7).$$

The operation  $\oplus$  has the following meaning:  $A \oplus B$  defines a probability distribution of events  $A$  and  $B$  in neighbor cells participated in a reaction caused the appearance of  $A \oplus B$ . Then  $\mathcal{A}$  simulates the Belousov-Zhabotinsky reaction.

**Definition 4** Let  $p, s_i, s_{i+1} \in \{Ce^{3+}, HBrO_2, BrO_3^-, H^+, Ce^{4+}, H_2O, BrCH(COOH)_2, Br^-, HCOOH, CO_2, HOBr, Br_2, CH_2(COOH)_2\}$ . A state  $p$  is called a premise for deducing  $s_{i+1}$  from  $s_i$  by the inference rule (1)  $\wedge$  (2)  $\wedge \dots \wedge$  (7) iff

- $p$  is  $s_i$  or
- in a neighbor cell we find out an expression of the form  $p \oplus A, B \oplus C$ , where  $A, B, C$  are propositional metavariables, i.e. they run over either the empty set or the set of states closed under the operation  $\oplus$ . Thus, we assume that each premise should occur in a separate cell. This means that if we find out an expression  $p_i \oplus p_j, B \oplus C$  or  $p_i \oplus A, p_j \oplus C$  in a neighbor cell and both  $p_i$  and  $p_j$  are needed for deducing, whereas  $p_i, p_j$  do not occur in other neighbor cells, then  $p_i, p_j$  could not be considered as premises.

$$x^{t+1}(z) = \begin{cases} (1) Ce^{4+} \oplus HBrO_2 \oplus H_2O, & \text{if } x^t(z) \in \{Ce^{3+}\} \text{ and} \\ & \text{premises } HBrO_2, BrO_3^-, H^+ \in (z + N); \\ (2) x^t(z), & \text{otherwise.} \end{cases} \quad (1)$$

$$x^{t+1}(z) = \begin{cases} (1) Br^- \oplus Ce^{3+} \oplus HCOOH \oplus CO_2 \oplus H^+, & \text{if } x^t(z) \in \{Ce^{4+}\} \\ & \text{and premises } BrCH(COOH)_2, H_2O \in (z + N); \\ (2) x^t(z), & \text{otherwise.} \end{cases} \quad (2)$$

$$x^{t+1}(z) = \begin{cases} (1) HOBr \oplus BrO_3^- \oplus H^+, & \text{if } x^t(z) \in \{HBrO_2\}; \\ (2) x^t(z), & \text{otherwise.} \end{cases} \quad (3)$$



$$x^{t+1}(z) = \begin{cases} (1) \text{ } HOBr \oplus HBrO_2, & \text{if } x^t(z) \in \{BrO_3^-\} \text{ and} \\ & \text{premises } Br^-, H^+ \in (z + N); \\ (2) \text{ } x^t(z), & \text{otherwise.} \end{cases} \quad (4)$$

$$x^{t+1}(z) = \begin{cases} (1) \text{ } HOBr, & \text{if } x^t(z) \in \{Br^-\} \text{ and} \\ & \text{premises } HBrO_2, H^+ \in (z + N); \\ (2) \text{ } x^t(z), & \text{otherwise.} \end{cases} \quad (5)$$

$$x^{t+1}(z) = \begin{cases} (1) \text{ } Br_2 \oplus H_2O, & \text{if } x^t(z) \in \{HOBr\} \text{ and} \\ & \text{premises } Br^-, H^+ \in (z + N); \\ (2) \text{ } x^t(z), & \text{otherwise.} \end{cases} \quad (6)$$

$$x^{t+1}(z) = \begin{cases} (1) \text{ } Br^- \oplus H^+ \oplus BrCH(COOH)_2, & \text{if } x^t(z) \in \{Br_2\} \\ & \text{and premises } CH_2(COOH)_2 \in (z + N); \\ (2) \text{ } x^t(z), & \text{otherwise.} \end{cases} \quad (7)$$

**Example 5 (Belousov-Zhabotinsky's cyclic proofs)** *We can simplify the automaton defined above assuming that  $\oplus$  is a metatheoretic operation with the following operational semantics:*

$$\frac{A \oplus B}{A} \quad \frac{A \oplus B}{B},$$

where  $A$  and  $B$  are metavariables defined on  $S$ . The informal meaning of that operation is that we can ignore one of both variables coupled by  $\oplus$ . In the cellular automaton  $\mathcal{A}$  this metaoperation will be used as follows:

$$x^{t+1}(z) = \begin{cases} X, Y, & \text{if } x^t(z) = A \oplus B \text{ and according to rules (1) - (7),} \\ & \text{X changes from A and Y changes from B;} \\ X, & \text{if } x^t(z) = A \oplus B \text{ and according to rules (1) - (7),} \\ & \text{X changes from A and B does not change;} \\ Y, & \text{if } x^t(z) = A \oplus B \text{ and according to rules (1) - (7),} \\ & \text{Y changes from B and A does not change;} \\ A \oplus B, & \text{if } x^t(z) = A \oplus B \text{ and rules (1) - (7)} \\ & \text{cannot be applied to A or B.} \end{cases} \quad (8)$$

Let us suppose now that  $X, Y$  run over the set of states closed under the operation  $\oplus$ .

$$x^{t+1}(z) = \begin{cases} X, Y, & \text{if (i) } x^t(z) = X, Y \text{ and (ii) both } X \text{ and } Y \\ & \text{are simultaneously usable (not usable)} \\ & \text{as premises in at least two different rules} \\ & \text{of (1) - (7) (see definition 4);} \\ X, & \text{if (i) } x^t(z) = X, Y \text{ and (ii) only } X \text{ is usable} \\ & \text{as a premise in at least one rule of (1) - (7);} \\ Y, & \text{if (i) } x^t(z) = X, Y \text{ and (ii) only } Y \text{ is usable} \\ & \text{as a premise in at least one rule of (1) - (7).} \end{cases} \quad (9)$$

$$\text{idempotency: } A ::= A, A. \quad (10)$$

$$\text{commutativity: } A, B ::= B, A. \quad (11)$$

Hence, we cannot ignore one of both variables coupled by  $\oplus$  and should accept both them if in the neighborhood there are reactants that catenate both variables and change them. This rule is the simplest interpretation of  $A \oplus B$  in definition 3. We have three cases: (i) both variables are catenated with reactants from the neighborhood, in this case we mean that the probability distribution of events  $A$  and  $B$  is the same and equal to 0.5 and, as a result, we cannot choose one of them and accept both; (ii) only  $A$  is catenated with reactants from the neighborhood, then the probability distribution of event  $A$  is equal to 1.0 and that of  $B$  to 0.0; (iii) only  $B$  is catenated with reactants from the neighborhood, then the probability distribution of event  $B$  is equal to 1.0 and that of  $A$  to 0.0. Thus,  $A \oplus B$  is a function that associates either exactly one value with its arguments (i.e. either  $A$  or  $B$ ) or simultaneously both values (i.e.  $A$  and  $B$ ).

This simplified version of the automaton  $\mathcal{A}$  is exemplified in Fig. 5.

Evidently, reducing the complicated dynamics of Belousov-Zhabotinsky reaction to conventional logical proofs is a task that cannot be solved in easy way differently from simulating within massive-parallel proofs.

## 4 Conclusion

In this paper we have considered a possibility of consistent proof theory in that there are no axioms or axiom schemata.

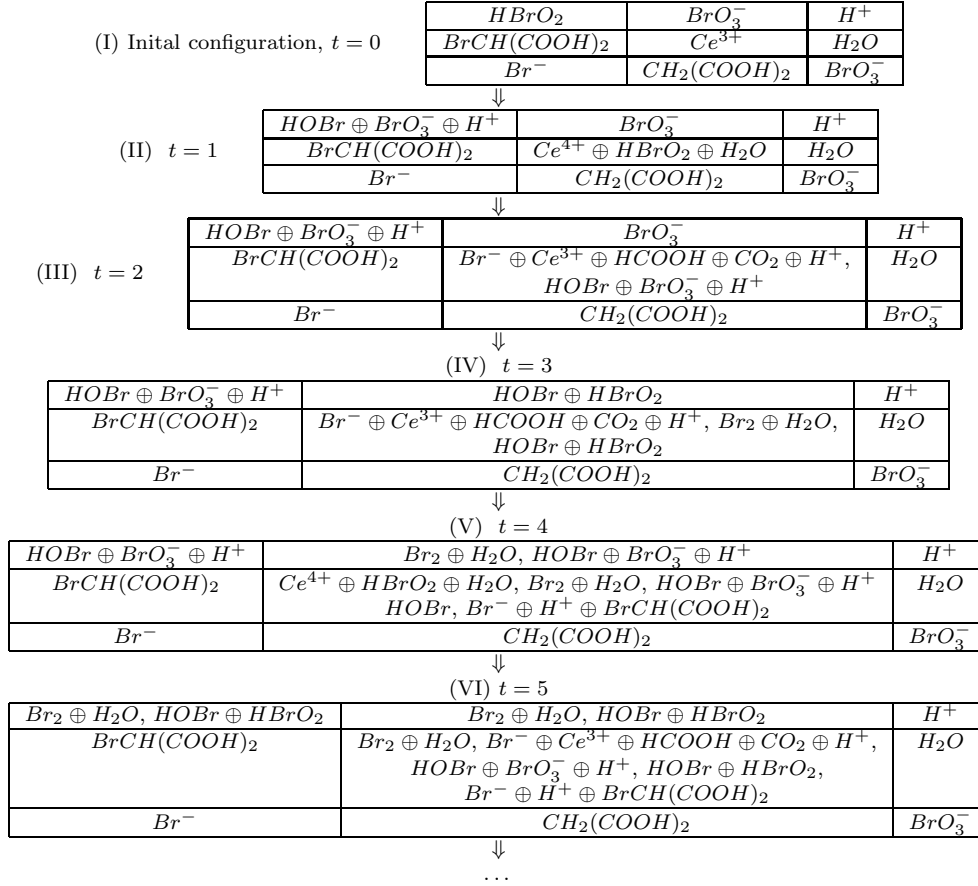


Figure 5: The evolution of a reversible proof-theoretic cellular automaton  $\mathcal{A}$  with the Moor neighborhood in the 2-dimensional space for the Belousov-Zhabotinsky reaction. This automaton simulates the circular feedback  $Ce^{3+} \rightarrow Ce^{4+} \rightarrow Ce^{3+} \rightarrow \dots$  (more precisely temporal oscillations in a well-stirred solution):  $Ce^{3+}$  is colorless and  $Ce^{4+}$  is yellow. The initial configuration of  $\mathcal{A}$ -cells described in (I) occurs in the same form at the further steps and the cycle repeats several times. For entailing (I)  $\rightarrow$  (II) we have just used inference rule (1) (row 2, column 2) and inference rule (3) (row 1, column 1), for entailing (II)  $\rightarrow$  (III) inference rules (2), (3) and (8) (row 2, column 2), for entailing (III)  $\rightarrow$  (IV) inference rule (4) (row 1, column 2) and inference rules (4), (6), (8) and (9) (row 2, column 2), for entailing (IV)  $\rightarrow$  (V) inference rules (3) and (6) (row 1, column 2) inference rules (1), (3), (5), (6), (7), (8) and (9) (row 2, column 2), for entailing (V)  $\rightarrow$  (VI) inference rules (4), (6) (row 1, column 1), inference rules (4), (6), (10) (row 1, column 2), inference rules (2), (3), (4), (6), (7), (9), (10), (11) (row 2, column 2).

## References

- [1] Adamatzky A. *Computing in Nonlinear Media and Automata Collectives*. Institute of Physics Publishing, 2001.
- [2] Adamatzky A., De Lacy Costello B., Asai T. *Reaction-Diffusion Computers*, Elsevier, 2005.
- [3] Adamatzky A., A. Wuensche, and B. De Lacy Costello, Glider-based computation in reaction-diffusion hexagonal cellular automata, *Chaos, Solitons & Fractals* 27, 2006, 287–295.
- [4] Berry G., Boudol G. The chemical abstract machine, *Teor. Comput. Sci.*, 96, 1992, 217–248.
- [5] Brotherston J. Cyclic proofs for first-order logic with inductive definitions [in:] B. Beckert, editor, *TABLEAUX 2005*, volume 3702 of LNAI, Springer-Verlag, 2005, 78–92.
- [6] Brotherston J. *Sequent Calculus Proof Systems for Inductive Definitions*. PhD thesis, University of Edinburgh, November 2006.
- [7] Brotherston J. Simpson, A., Complete sequent calculi for induction and infinite descent. *LICS-22*, IEEE Computer Society, July 2007, 51–60.
- [8] Ivanitsky G. R., Kunisky A. S., Tzyganov M. A. Study of ‘target patterns’ in a phage-bacterium system, *Self-organization: Autowaves and Structures Far From Equilibrium*. Ed. V.I. Krinsky. Heidelberg-Springer, 1984, 214–217.
- [9] Khrennikov A., Schumann A. Physics Beyond The Set-Theoretic Axiom of Foundation, [in:] *AIP Conf. Proc.* – March 10, 2009 – Volume 1101. 374–380.
- [10] Prajer M., Fleury A., Laurent M. Dynamics of calcium regulation in Paramecium and possible morphogenetic implication, *Journal Cell Sci.*, 110, 1997, 529–535.
- [11] Santocanale L., A calculus of circular proofs and its categorical semantics, [in:] M. Nielsen and U. Engberg, editors, *Proc. of FoSSaCS 2002*, Grenoble, Apr. 2002, Springer-Verlag LNCS 2303, 357–371.
- [12] Schumann A., Adamatzky A. Towards Semantical Model of Reaction-Diffusion Computing, *Kybernetes*, 38 (9), 2009, pp. 1518 - 1531.

- [13] Schumann A., Adamatzky A. Physarum Spatial Logic, *New Mathematics and Natural Computation*, 2010 (to appear).
- [14] Schumann A. Non-well-founded probabilities on streams, [in:] D. Dubois et al., editors, *Soft Methods for Handling Variability and Imprecision*, Advances in Soft Computing 48, 2008, 59–65.

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